Non-Gaussianity of Large-Scale Cosmic Microwave Background Anisotropies beyond Perturbation Theory

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(Dated: February 2, 2008)

Abstract

We compute the fully non-linear Cosmic Microwave Background (CMB) anisotropies on scales larger than the horizon at last-scattering in terms of only the curvature perturbation, providing a generalization of the linear Sachs-Wolfe effect at any order in perturbation theory. We show how to compute the n-point connected correlation functions of the large-scale CMB anisotropies for generic primordial seeds provided by standard slow-roll inflation as well as the curvaton and other scenarios for the generation of cosmological perturbations. As an application of our formalism, we compute the three- and four-point connected correlation functions whose detection in future CMB experiments might be used to assess the level of primordial non-Gaussianity, giving the theoretical predictions for the parameters of quadratic and cubic non-linearities $f_{\rm NL}$ and $g_{\rm NL}$.

PACS numbers: 98.80.cq

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I. INTRODUCTION

Cosmological inflation [1] is the dominant paradigm to understand the initial conditions for CMB anisotropies and structure formation. In the inflationary picture, the primordial cosmological perturbations are created from quantum fluctuations "redshifted" out of the horizon during an early period of superluminal expansion of the universe, where they remain "frozen". They are observable as temperature anisotropies in the CMB at the last scattering surface. They were first detected by the Cosmic Background Explorer (COBE) satellite [2, 3, 4]. The last and most impressive confirmation of the inflationary paradigm has been recently provided by the data of the Wilkinson Microwave Anisotropy Probe (WMAP) mission [5]. Since the observed cosmological perturbations are of the order of 10^{-5}, one might think that first-order perturbation theory will be adequate for all comparison with observations. That may not be the case however, as the Planck satellite [6] and its successors may be sensitive to the non-Gaussianity of the cosmological perturbations at the level of second- or higher-order perturbation theory [7]. Statistics like the bispectrum and the trispectrum of the CMB can be used to assess the level of primordial non-Gaussianity on various cosmological scales and to discriminate it from the one induced by secondary anisotropies and systematic effects [7, 8, 9, 10]. Therefore, it is of fundamental importance to provide accurate theoretical predictions for the statistics of the large-angle CMB anisotropies as left imprinted by the primordial seeds originated during or immediately after inflation. Steps towards this goal have been taken in Refs. [11, 12, 13, 14] at the level of second order perturbation theory.

In this paper we derive an expression for the anisotropies of the CMB on scales larger than the horizon at last scattering which is valid at any order in perturbation theory, providing a fully non-linear generalization to the Sachs-Wolfe effect at first- [15] and second-order [13]. In particular, for the standard single-field models of inflation, we provide the exact non-perturbative expression for the bispectrum and the trispecturm in the so-called "squeezed" limit in which some of the wavenumbers are much smaller than the others. Furthermore, we compute the generic expressions for the non-linearity parameters $f_{\rm NL}$ and $g_{\rm NL}$ characterizing respectively the quadratic and cubic non-linearity in the large-angle CMB anisotropies.

The paper is organized as follows. In Section II we provide the non-linear generalization of the Sachs-Wolfe effect which is expressed in terms of the comoving curvature perturbation from inflation in section III. In Section IV we show how to compute the connected n-point correlation functions of the CMB anisotropies, leaving the details to the Appendix. Finally, we draw our conclusions in Section V.

II. THE NON-LINEAR SACHS-WOLFE EFFECT

Our starting point is the Arnowitt-Deser-Misner (ADM) formalism which is particularly useful to deal with the non-linear evolution of cosmological perturbations. The line element is

$$ds^2 = -N^2\, dt^2 + N_i\, dt\, dx^i + \gamma_{ij}\, dx^i\, dx^j\,, \tag{1}$$

where the three-metric γ_{ij}, the lapse N and the shift N_i functions describe the evolution of timelike hypersurfaces. In the ADM formalism the equations simplify considerably if we set $N^i = 0$. Moreover we are interested only in scalar perturbations in a flat Universe and therefore we find it convenient to recast the metric as

$$ds^2 = -e^{2\Phi}\, dt^2 + a^2(t) e^{-2\Psi} \delta_{ij}\, dx^i\, dx^j\,, \tag{2}$$

where $a(t)$ is the scale factor describing the evolution of the homogeneous and isotropic Universe and we have introduced two gravitational potentials Φ and Ψ. The expression (2) holds at any order in perturbation theory. To make contact with the usual perturbative approach, one may expand the gravitational potentials at first- and second-order, *e.g*, $\Phi = \Phi_1 + \Phi_2/2$. From Eq. (2) one recovers at linear order the well-known longitudinal gauge, $N^2 = (1+2\Phi_1)$ and $\gamma_{ij} = a^2(1-2\Psi_1)\delta_{ij}$. At second-order, one finds $\Phi_2 = \phi_2 - 2\phi_1^2$ and $\Psi_2 = \psi_2 + 2\psi_1^2$ where ϕ_1, ψ_1 and ϕ_2, ψ_2 (with $\phi_1 = \Phi_1$ and $\psi_1 = \Psi_1$) are the first and second-order gravitational potentials in the longitudinal (Poisson) gauge adopted in Refs. [7, 16], $N^2 = (1 + 2\phi_1 + \phi_2)$ and $\gamma_{ij} = a^2(1 - 2\psi_1 - \psi_2)\delta_{ij}$ as far as scalar perturbations are concerned. In writing Eq. (2) we have neglected vector and tensor perturbation modes. For the vector perturbations the reason is that we are interested in long-wavelength perturbations, *i.e.* on scales larger than the horizon at last scattering, while vector modes will contain gradient terms being produced as non-linear combination of scalar-modes and thus they will be more important on small scales (linear vector modes are not generated in standard mechanisms for cosmological perturbations, as inflation). For example the results of Ref. [16] show clearly this for second-order perturbations. The tensor contribution can be neglected for two reasons. First, the tensor perturbations produced from inflation on large scales give a negligible contribution to the higher-order statistics of the Sachs-Wolfe effect being of the order of (powers of) the slow-roll parameters during inflation (this holds for linear tensor modes as well as for tensor modes generated by the non-linear evolution of scalar perturbations during inflation, for example see the results of Ref. [24] for second-order perturbations). Moreover, while on large scales the tensor modes have been proven to remain constant in time [17], when they approach the horizon they have a wavelike contribution which oscillates with decreasing amplitude.

Since we are interested in the cosmological perturbations on large scales, that is in perturbations whose wavelength is larger than the Hubble radius at last scattering, a local observer would see them in the form of a classical – possibly time-dependent – (nearly zero-momentum) homogeneous and isotropic background. Therefore, it should be possible to perform a change of coordinates in such a way as to absorb the super-Hubble modes and work with a metric of an homogeneous and isotropic Universe (plus, of course, cosmological perturbations on scale smaller than the horizon). We split the gravitational potential Φ as

$$\Phi = \Phi_\ell + \Phi_s \,, \tag{3}$$

where Φ_ℓ stands for the part of the gravitational potential receiving contributions only from the super-Hubble modes; Φ_s receives contributions only from the sub-horizon modes

$$\begin{aligned} \Phi_\ell &= \int \frac{d^3k}{(2\pi)^3}\, \theta\left(aH - k\right)\, \Phi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} \,, \\ \Phi_s &= \int \frac{d^3k}{(2\pi)^3}\, \theta\left(k - aH\right)\, \Phi_{\vec{k}}\, e^{i\vec{k}\cdot\vec{x}} \,, \end{aligned} \tag{4}$$

where H is the Hubble rate computed with respect to the cosmic time, $H = \dot{a}/a$, and $\theta(x)$ is the step function. Analogous definitions hold for the other gravitational potential Ψ.

By construction Φ_ℓ and Ψ_ℓ are a collection of Fourier modes whose wavelengths are larger than the horizon length and we may safely neglect their spatial gradients. Therefore Φ_ℓ and Ψ_ℓ are only functions of time. This amounts to saying that we can absorb the large-scale perturbations in the metric (2) by the following redefinitions

$$d\overline{t} \;=\; e^{\Phi_\ell} dt \,, \tag{5}$$

$$\overline{a} \;=\; a\, e^{-\Psi_\ell} \,. \tag{6}$$

The new metric describes a homogeneous and isotropic Universe

$$ds^2 = -d\overline{t}^2 + \overline{a}^2 \delta_{ij}\, dx^i\, dx^j \,, \tag{7}$$

where for simplicity we have not included the sub-horizon modes. On super-horizon scales one can regard the Universe as a collection of regions of size of the Hubble radius evolving like unperturbed patches with metric (7) [17].

Let us now go back to the quantity we are interested in, namely the anisotropies of the CMB as measured today by an observer $\mathcal{O}$. If she/he is interested in the CMB anisotropies at large scales, the effect of super-Hubble modes is encoded in the metric (7). During their travel from the last scattering surface – to be considered as the emitter point $\mathcal{E}$ – to the observer, the CMB photons suffer a redshift determined by the ratio of the emitted frequency $\overline{\omega}_{\mathcal{E}}$ to the observed one $\overline{\omega}_{\mathcal{O}}$

$$\overline{T}_{\mathcal{O}} = \overline{T}_{\mathcal{E}}\, \frac{\overline{\omega}_{\mathcal{O}}}{\overline{\omega}_{\mathcal{E}}} \,, \tag{8}$$

where $\overline{T}_{\mathcal{O}}$ and $\overline{T}_{\mathcal{E}}$ are the temperatures at the observer point and at the last scattering surface, respectively.

What is then the temperature anisotropy measured by the observer? The expression (8) shows that the measured large-scale anisotropies are made of two contributions: the intrinsic inhomogeneities in the temperature at the last scattering surface and the inhomogeneities in the scaling factor provided by the ratio of the frequencies of the photons at the departure and arrival points. Let us first consider the second contribution. As the frequency of the photon is the inverse of a time period, we get immediately the fully non-linear relation

$$\frac{\overline{\omega}_{\mathcal{E}}}{\overline{\omega}_{\mathcal{O}}} = \frac{\omega_{\mathcal{E}}}{\omega_{\mathcal{O}}} e^{-\Phi_{\ell\mathcal{E}} + \Phi_{\ell\mathcal{O}}} \,. \tag{9}$$

As for the temperature anisotropies coming from the intrinsic temperature fluctuation at the emission point, it maybe worth to recall how to obtain this quantity in the longitudinal gauge at first order. By expanding the photon energy density $\rho_\gamma \propto T_\gamma^4$, the intrinsic temperature anisotropies at last scattering are given by $\delta_1 T_{\mathcal{E}}/T_{\mathcal{E}} = (1/4)\delta_1\rho_\gamma/\rho_\gamma$. One relates the photon energy density fluctuation to the gravitational perturbation first by implementing the adiabaticity condition $\delta_1\rho_\gamma/\rho_\gamma = (4/3)\delta_1\rho_m/\rho_m$, where $\delta_1\rho_m/\rho_m$ is the relative fluctuation in the matter component, and then using the energy constraint of Einstein equations $\Phi_1 = -(1/2)\delta_1\rho_m/\rho_m$. The result is $\delta_1 T_{\mathcal{E}}/T_{\mathcal{E}} = -2\Phi_{1\mathcal{E}}/3$. Summing this contribution to the anisotropies coming from the redshift factor (9) expanded at first order provides the standard (linear) Sachs-Wolfe effect $\delta_1 T_{\mathcal{O}}/T_{\mathcal{O}} = \Phi_{1\mathcal{E}}/3$. Following the same steps, we may easily obtain its full non-linear generalization.

Let us first relate the photon energy density $\overline{\rho}_\gamma$ to the energy density of the non-relativistic matter $\overline{\rho}_m$ by using the adiabaticity conditon. Again here a bar indicates that we are considering quantities in the locally homogeneous Universe described by the metric (7). Using the energy continuity equation on large scales $\partial\overline{\rho}/\partial\overline{t} = -3\overline{H}(\overline{\rho}+\overline{P})$, where $\overline{H} = d\ln\overline{a}/d\overline{t}$ and $\overline{P}$ is the pressure of the fluid, one can easily show that there exists a conserved quantity in time at any order in perturbation theory [19]

$$\mathcal{F} \equiv \ln\overline{a} + \frac{1}{3}\int^{\overline{\rho}} \frac{d\overline{\rho'}}{\left(\overline{\rho'}+\overline{P'}\right)}\,. \tag{10}$$

The perturbation $\delta\mathcal{F}$ is a gauge-invariant quantity representing the non-linear extension of the curvature perturbation ζ on uniform energy density hypersurfaces on superhorizon scales for adiabatic fluids [19]. Indeed, expanding it at first and second order one gets the corresponding definition $\zeta_1 = -\psi_1 - \delta_1\rho/\dot{\rho}$ and the quantity ζ_2 introduced in Ref. [20]. At first order the adiabaticity condition corresponds to set $\zeta_{1\gamma} = \zeta_{1m}$ for the curvature perturbations relative to each component. At the non-linear level the adiabaticity condition generalizes to

$$\frac{1}{3}\int\frac{d\overline{\rho}_m}{\overline{\rho}_m} = \frac{1}{4}\int\frac{d\overline{\rho}_\gamma}{\overline{\rho}_\gamma}\,, \tag{11}$$

or

$$\ln\overline{\rho}_m = \ln\overline{\rho}_\gamma^{3/4}\,. \tag{12}$$

To make contact with the standard second-order result, we may expand in Eq. (12) the photon energy density perturbations as $\delta\overline{\rho}_\gamma/\rho_\gamma = \delta_1\rho_\gamma/\rho_\gamma + \frac{1}{2}\delta_2\rho_\gamma/\rho_\gamma$, and similarly for the matter component. We immediately recover the adiabaticity condition

$$\frac{\delta_2\rho_\gamma}{\rho_\gamma} = \frac{4}{3}\frac{\delta_2\rho_m}{\rho_m} + \frac{4}{9}\left(\frac{\delta_1\rho_m}{\rho_m}\right)^2 \tag{13}$$

given in Ref. [7].

Next we need to relate the photon energy density to the gravitational potentials at the non-linear level. The energy constraint inferred from the (0-0) component of Einstein equations in the matter-dominated era with the "barred" metric (7) is

$$\overline{H}^2 = \frac{8\pi G_N}{3}\overline{\rho}_m\,. \tag{14}$$

Using Eqs. (5) and (6) the Hubble parameter $\overline{H}$ reads

$$\overline{H} = \frac{1}{\overline{a}}\frac{d\overline{a}}{d\overline{t}} = e^{-\Phi_\ell}(H - \dot{\Psi}_\ell)\,, \tag{15}$$

where $H = d\ln a/dt$ is the Hubble parameter in the "unbarred" metric. Eq. (14) thus yields an expression for the energy density of the non-relativistic matter which is fully nonlinear, being expressed in terms of the gravitational potential Φ_ℓ

$$\overline{\rho}_m = \rho_m e^{-2\Phi_\ell}\,, \tag{16}$$

where we have dropped $\dot{\Psi}_\ell$ which is negligible on large scales. By perturbing the expression (16) we are able to recover in a straightforward way the solutions of the (0-0) component of Einstein equations for a matter-dominated Universe in the large-scale limit obtained at second-order in perturbation theory. Indeed, recalling that Φ is perturbatively related to the quantity $\phi = \phi_1 + \phi_2/2$ used in Ref. [7] by $\Phi_1 = \phi_1$ and $\Phi_2 = \phi_2 - 2(\phi_1)^2$, one immediately obtains [7, 21]

$$\begin{aligned} \frac{\delta_1\rho_m}{\rho_m} &= -2\phi_1\,, \\ \frac{1}{2}\frac{\delta_2\rho_m}{\rho_m} &= -\phi_2 + 4(\phi_1)^2\,. \end{aligned} \tag{17}$$

The expression for the intrinsic temperature of the photons at the last scattering surface $\overline{T}_{\mathcal{E}} \propto \overline{\rho}_\gamma^{1/4}$ follows from Eqs. (12) and (16)

$$\overline{T}_{\mathcal{E}} = T_{\mathcal{E}}\, e^{-2\Phi_\ell/3}\,. \tag{18}$$

Plugging Eqs. (9) and (18) into the expression (8) we are finally able to provide the expression for the CMB temperature which is fully nonlinear and takes into account both the gravitational redshift of the photons due to the metric perturbations at last scattering and the intrinsic temperature anisotropies

$$\overline{T}_{\mathcal{O}} = \left(\frac{\omega_{\mathcal{O}}}{\omega_{\mathcal{E}}}\right) T_{\mathcal{E}}\, e^{\Phi_\ell/3}\,. \tag{19}$$

From Eq. (19) we read the *non-perturbative* anisotropy corresponding to the Sachs-Wolfe effect

$$\frac{\delta_{np}\overline{T}_{\mathcal{O}}}{T_{\mathcal{O}}} = e^{\Phi_\ell/3} - 1\,. \tag{20}$$

Eq. (20) is one of the main results of this paper and represents *at any order in perturbation theory* the extension of the linear Sachs-Wolfe effect. At first order one gets

$$\frac{\delta_1 T_{\mathcal{O}}}{T_{\mathcal{O}}} = \frac{1}{3}\Phi_1\,, \tag{21}$$

and at second order

$$\frac{1}{2}\frac{\delta_2 T_{\mathcal{O}}}{T_{\mathcal{O}}} = \frac{1}{6}\Phi_2 + \frac{1}{18}\left(\Phi_1\right)^2\,, \tag{22}$$

which exactly reproduces the generalization of the Sachs-Wolfe effect at second-order in the perturbations found in Ref. [7, 21] (where $\Phi_1 = \phi_1$ and $\Phi_2 = \phi_2 - 2(\phi_1)^2$).

III. RELATING THE CMB ANISOTROPIES TO THE INFLATIONARY COMOVING CURVATURE PERTURBATION

In this section we relate the gravitational potentials Φ_ℓ to Ψ_ℓ to the curvature perturbation $\zeta = \delta\mathcal{F}$ at any order in perturbation theory (for notational simplicity we drop the subscrip "ℓ" from now on). This will allow to express the non-linear temperature fluctuations in terms of the initial conditions provided by inflation. We use the evolution equation in the ADM formalism

$$-N^{|i}{}_{|k} + \frac{1}{3}N^{|l}{}_{|l}\delta^i{}_k + N\ {}^{(3)}\overline{R}^i{}_k = N\ 8\pi G\ \overline{S}^i{}_k\,, \tag{23}$$

where a vertical bar denotes a covariant derivative with respect to γ_{ij} and we have used that fact that the traceless part of the extrinsic curvature $\overline{K}_{ij} = K_{ij} - K\,\gamma_{ij}/3$ vanishes in our metric (7). The extrinsic curvature is defined as $K_{ij} = -\dot{\gamma}_{ij}/(2N)$ and $K = \gamma^{ij}K_{ij}$. Here ${}^{(3)}\overline{R}^i{}_k = \gamma^{ij}\ {}^{(3)}\overline{R}_{jk} ={}^{(3)} R^i_k -{}^{(3)} R\ \delta^i_k/3$ where ${}^{(3)}R_{jk}$ is the Ricci tensor of constant time hypersurfaces associated with the metric γ_{ij}. Analogous definitions hold for $\overline{S}^i{}_k$ constructed from the matter stress-energy three-tensor $S_{ij} = T_{ij}$.

From Eq. (23) we want to obtain a constraint between the gravitational potentials Φ and Ψ keeping track of non-local terms. Thus we cannot neglect the gradient terms appearing in this equation. Since $N = e^\Phi$ we find

$$\begin{aligned}
N^{|i}{}_{|k} &= \frac{e^{2\Psi}e^{\Phi}}{a^2(t)}\Big(\Phi_{,k}\Phi^{,i} + \Phi^{,i}{}_{,k} + \Psi_{,k}\Phi^{,i} + \Psi^{,i}\Phi_{,k} - \Psi_{,j}\Phi^{,j}\delta^i{}_k\Big)\,,\\
{}^{(3)}R^i{}_k &= \frac{e^{2\Psi}}{a^2(t)}\Big(\Psi^{,i}{}_{,k} + \nabla^2\Psi\delta^i{}_k + \Psi^{,i}\Psi_{,k} - \Psi_{,m}\Psi^{,m}\delta^i{}_k\Big)\,,\\
{}^{(3)}R &= 2\frac{e^{2\Psi}}{a^2(t)}\left(2\nabla^2\Psi - \Psi_{,i}\Psi^{,i}\right)\,.
\end{aligned} \tag{24}$$

Hence Eq. (23) reads

$$
\begin{aligned}
&- \Big(\Phi_{,k}\Phi^{,i} + \Phi^{,i}_{\ \ ,k} + \Psi_{,k}\Phi^{,i} + \Psi^{,i}\Phi_{,k} - \Psi_{,j}\Phi^{,j}\delta^i_{\ k}\Big) + \frac{1}{3}(\nabla^2\Phi + \Phi^{,l}\Phi_{,l} - \Psi_{,l}\Phi^{,l}\Big)\delta^i_{\ k} \\
&+ \Big(\Psi^{,i}_{\ \ ,k} + \nabla^2\Psi\delta^i_{\ k} + \Psi^{,i}\Psi_{,k} - \Psi_{,m}\Psi^{,m}\delta^i_{\ k}\Big) - \frac{2}{3}\Big(2\nabla^2\Psi - \Psi_{,l}\Psi^{,l}\Big)\delta^i_{\ k} = 8\pi G\, a^2(t)e^{-2\Psi}\overline{S}^i_{\ k}\,,
\end{aligned}
\tag{25}
$$

where the indices of the partial derivatives are raised by δ^{ij}. Notice that at first order this equation gives the usual constraint $\Phi_1 = \Psi_1$. As far as the matter content is concerned we can consider the perfect fluid energy momentum tensor $T_{\mu\nu} = \left(\overline{\rho} + \overline{P}\right) u_\mu u_\nu + \overline{P}\, g_{\mu\nu}$ and we find

$$
\begin{aligned}
S^i_{\ k} &= \gamma^{ij}T_{jk} = a^{-2}(t)e^{2\Psi}\left(\overline{\rho} + \overline{P}\right) u^i u_k + \overline{P}\delta^i_{\ k}\,, \\
\overline{S}^i_{\ k} &= a^{-2}(t)e^{2\Psi}\left(\overline{\rho} + \overline{P}\right)(u^i u_k - u^i u_i \delta^i_{\ k}/3)\,,
\end{aligned}
\tag{26}
$$

where $u^i = \delta^{ij}u_j$. We need an expression for the spatial velocities. Thus we use the momentum constraint of the ADM equations which reads [22]

$$
8\pi G\, J_i = -\frac{2}{3}K_{|i}\,,
\tag{27}
$$

where $J_i = NT^0_{\ i} = e^{\Phi}T^0_{\ i} = -e^{-\Phi}T_{0i}$ is the momentum density and $K = -3e^{-\Phi}(H(t) - \dot{\Psi})$. One can use the normalization $g^{\mu\nu}u_\mu u_\nu = -1$ to express u_0, finding $u_0 = -(1 + a^{-2}(t)e^{2\Psi}u^i u_i)^{1/2}$. Let us just consider scalar velocities $u^i = \partial_i u$ (on sufficiently large scales vector modes may be safely neglected). From Eq. (27) we obtain

$$
(\overline{\rho} + \overline{P})u_i(1 + a^{-2}(t)e^{2\Psi}u^i u_i)^{1/2} = \Big[\frac{e^{-\Phi}}{4\pi G}(H(t) - \dot{\Psi})\Big]_{|i}
\tag{28}
$$

In fact we are interested in the case of non-relativistic matter $\overline{P}_m = 0$ and $\overline{\rho}_m = \rho_m e^{-2\Phi}$. As Eq. (25) contains at least two gradients (also $\overline{S}^i_k$ contains at least two spatial gradients), using the gradient expansion we may restrict ourselves to the solution

$$
u_i = \frac{e^{2\Phi}}{4\pi G\,\rho_m}\Big[e^{-\Phi}(H(t) - \dot{\Psi})\Big]_{|i} = -\frac{e^{\Phi}H(t)}{4\pi G\,\rho_m}\Phi_{,i}\,,
\tag{29}
$$

where we have neglected $\dot{\Psi}$. We thus find

$$
\overline{S}^i_{\ k} = \frac{e^{2\Psi}}{6\pi G\, a^2}\Big(\Phi^{,i}\Phi_{,k} - \frac{1}{3}\Phi^{,j}\Phi_{,j}\delta^i_{\ k}\Big)\,.
\tag{30}
$$

Inserting Eq. (30) into Eq (25) and applying the operator $\partial_i\partial^k$ we find

$$\begin{aligned}
\nabla^4\left(\Psi-\Phi\right) &= -\frac{3}{2}\partial_i\partial^k\left(\Psi^{,i}\Psi_{,k}\right)+\frac{1}{2}\nabla^2\left(\Psi^{,i}\Psi_{,i}\right) \\
&+ \frac{7}{2}\partial_i\partial^k\left(\Phi^{,i}\Phi_{,k}\right)-\frac{7}{6}\nabla^2\left(\Phi^{,i}\Phi_{,i}\right) \\
&+ 3\partial_i\partial^k\left(\Phi^{,i}\Psi_{,k}\right)-\nabla^2\left(\Phi^{,i}\Psi_{,i}\right)\,. \qquad (31)
\end{aligned}$$

Notice that Eq. (31) is the non-linear generalization of the constraint between the gravitational potentials Φ and Ψ in the longitudinal (Poisson) gauge and is valid at any order in perturbation theory. At linear order one recovers the well-known result $\Phi_1 = \Psi_1$, while at second order the relation first found in Refs. [7, 18] follows $\psi_2 - \phi_2 = -4\psi_1^2 + 10\nabla^{-4}\partial_i\partial^k(\psi_1^{,i}\psi_{1,k}) - \frac{10}{3}\nabla^{-2}(\psi_1^{,i}\psi_{1,i})$ (where $\Phi_2 = \phi_2 - 2\phi_1^2$ and $\Psi_2 = \psi_2 + 2\psi_1^2$ and $\phi_1 = \Phi_1$, $\psi_1 = \Psi_1$).

In the following we find it convenient to write the relation (31) as $\Psi = \Phi + \mathcal{K}[\Phi, \Psi]$ where the kernel $\mathcal{K}$ is obtained by acting through the operator ∇^{-4} onto the *r.h.s.* of Eq. (31) and takes into account non-local terms. Going to momentum space, one easily realizes that in the specific "squeezed" limit of Ref. [24], where one of the wavenumbers is much smaller than the other two, *e.g.* $k_1 \ll k_{2,3}$, the kernel $\mathcal{K} \to 0$. From Eq. (10) the curvature perturbation is given by

$$\zeta \equiv \delta\mathcal{F} = -\Psi + \frac{1}{3}\ln\frac{\overline{\rho}}{\rho} = -\Psi - \frac{2}{3}\Phi\,, \qquad (32)$$

where we have used $\overline{\rho}_m = \rho_m e^{-2\Phi}$. Hence we finally find $-5\Phi/3 = \zeta + \mathcal{K}$ and we can recast the non-linear temperature anisotropies (20) as

$$\frac{\delta_{np}\overline{T}_{\mathcal{O}}}{T_{\mathcal{O}}} = e^{-\zeta/5-\mathcal{K}/5} - 1\,, \qquad (33)$$

which is the starting point to evaluate the $n-$point correlation function for CMB temperature anisotropies. The comoving curvature perturbation ζ is conserved at any order in perturbation theory on large scales and therefore one can fix its properties right at the end of inflation. Since for standard inflation the curvature perturbation can be considered as a Gaussian distributed quantity [23, 24] (deviations from non-Gaussianity are proportional to deviations of the spectral index from unity and are, therefore, tiny), we will adopt ζ as our Gaussian seed in the case of standard single-field models of inflation.

IV. CORRELATION FUNCTIONS OF LARGE-SCALE CMB ANISOTROPIES

As a warm-up excercise, let us first consider a simpler case and evaluate the n-point correlation function of $e^{\varphi(\mathbf{x})}$ where $\varphi(\mathbf{x})$ is a Gaussian random field. Applying the well-known techniques of quantum field theory, it turns out that

$$\left\langle e^{\varphi(\mathbf{x}_1)} \ldots\ldots e^{\varphi(\mathbf{x}_N)} \right\rangle = e^{\frac{1}{2}\int d\mathbf{x} d\mathbf{y}\, J(\mathbf{x})\, \langle \varphi(\mathbf{x})\varphi(\mathbf{y})\rangle\, J(\mathbf{y})} , \tag{34}$$

where $J(\mathbf{x}) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_i)$ corresponds in fact to the source term appearing in the path integral formulation. The calculation of the integral in Eq. (34) brings

$$\left\langle e^{\varphi(\mathbf{x}_1)} \ldots\ldots e^{\varphi(\mathbf{x}_N)} \right\rangle = e^{\frac{1}{2}\sum_{i,j}\langle \varphi(\mathbf{x}_i)\varphi(\mathbf{x}_j)\rangle} . \tag{35}$$

Notice, for example, that for the 2-point correlation function we find the usual result

$$\left\langle e^{\varphi(\mathbf{x}_1)} e^{\varphi(\mathbf{x}_2)} \right\rangle = e^{\langle \varphi^2 \rangle} e^{\langle \varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)\rangle} , \tag{36}$$

which for $\mathbf{x}_1 = \mathbf{x}_2$ gives $\langle e^{2\varphi(\mathbf{x})}\rangle = e^{2\langle\varphi^2\rangle}$, where $\langle\varphi^2\rangle = \langle\varphi^2(\mathbf{x})\rangle$.

For the 3-point function one finds

$$\left\langle e^{\varphi(\mathbf{x}_1)} e^{\varphi(\mathbf{x}_2)} e^{\varphi(\mathbf{x}_3)} \right\rangle = e^{\frac{3}{2}\langle\varphi^2\rangle}\, e^{\langle\varphi_1\varphi_2\rangle + \langle\varphi_1\varphi_3\rangle + \langle\varphi_2\varphi_3\rangle} , \tag{37}$$

where for simplicity we have used the notation $\langle\varphi_i\varphi_j\rangle = \langle\varphi(\mathbf{x}_i)\varphi(\mathbf{x}_j)\rangle$. If now we expand the exponential in the limit in which the two point function is small we obtain

$$\begin{aligned} \left\langle e^{\varphi(\mathbf{x}_1)} e^{\varphi(\mathbf{x}_2)} e^{\varphi(\mathbf{x}_3)} \right\rangle \;\simeq\; & e^{\frac{3}{2}\langle\varphi^2\rangle}\Big[1 + \Big(\langle\varphi_1\varphi_2\rangle + \text{cycl.}\Big) \\ & + \frac{1}{2}\Big(\langle\varphi_1\varphi_2\rangle^2 + \text{cycl.}\Big) \\ & + \Big(\langle\varphi_1\varphi_2\rangle\langle\varphi_1\varphi_3\rangle + \text{cycl.}\Big)\Big] . \end{aligned} \tag{38}$$

It is in fact a term analogous to the last contribution in Eq. (38) that enters in the three-point function of the CMB anisotropies (20) if we are interested in non-linearities up to second-order terms only. The complication arises from the fact that the gravitational potential Φ appearing in Eq. (20) is not a Gaussian variable, since it can already contain quadratic (and higher order) terms in the Gaussian variable ζ in the case of single-field models of inflation. These will add non-Gaussian contributions which are contained in the kernel $\mathcal{K}$. In other scenarios for the generation of the cosmological perturbations ζ is not a Gaussian quantity and we will properly take into account the primordial non-Gaussian contributions.

Let us see how to generalize the previous procedure to this case. First of all we apply an iterative procedure to express the kernel $\mathcal{K}$ in terms of powers of ζ. We will use Eqs. (31) and (32). For the zeroth and first-order terms of the iteration we find (the suffix does not refer to the order of the expansion in the perturbations, but to the order of the approximation given by the iteration procedure: each r-th term contains up to $(r+1)$ powers of ζ)

$$\Phi^{(0)} = -\frac{3}{5}\zeta \tag{39}$$

$$\Phi^{(1)} = -\frac{3}{5}\zeta - \left(\frac{3}{5}\right)^3 \mathcal{K}[\zeta^2]\,, \tag{40}$$

and for the next terms $(n = 1, 2, ..)$

$$\begin{aligned}\Phi^{(2n)} &= \Phi^{(2n-1)} + \mathcal{K}_1[\Phi^{(0)}, \Phi^{(2n-2)} - \Phi^{(2n-1)}] \\ &+ \sum_{m=0}^{n-2} \mathcal{K}_1[\Phi^{(m)} - \Phi^{(m+1)}, \Phi^{(2n-m-3)} - \Phi^{(2n-m-2)}]\,, \end{aligned} \tag{41}$$

$$\begin{aligned}\Phi^{(2n+1)} &= \Phi^{(2n)} + \mathcal{K}_1[\Phi^{(0)}, \Phi^{(2n-1)} - \Phi^{(2n)}] \\ &+ \sum_{m=0}^{n-2} \mathcal{K}_1[\Phi^{(m)} - \Phi^{(m+1)}, \Phi^{(2n-m-2)} - \Phi^{(2n-m-1)}] \\ &+ \mathcal{K}_2[(\Phi^{(n-1)} - \Phi^{(n)})^2]\,, \end{aligned} \tag{42}$$

where we have introduced the bilinear operators

$$\begin{aligned}\mathcal{K}_1[(\cdot),(\cdot)] &\equiv \nabla^{-4}(6\partial_i\partial^k[(\cdot)^{,i}(\cdot)_{,k}] - 2\nabla^2[(\cdot)^{,i}(\cdot)_{,i}])\,, \\ \mathcal{K}_2[(\cdot),(\cdot)] &\equiv -\nabla^{-4}\left(\frac{1}{2}\partial_i\partial^k\left[(\cdot)^{,i}(\cdot)_{,k}\right] - \frac{1}{2}\nabla^2[(\cdot)^{,i}(\cdot)_{,i}]\right)\,. \end{aligned} \tag{43}$$

Notice that for equal entries $\mathcal{K}_1 = 6\mathcal{K}/5$. If the upper limit of the sums appearing in these expressions turns out to be negative the sum must be taken to be vanishing.

Thus we can use Eq. (39) and (40) to find the next order approximations for the expression of Φ in terms of the Gaussian curvature perturbation ζ. For example for $\Phi^{(2)}$ up to $\mathcal{O}(\zeta^3)$ contributions we have

$$\Phi^{(2)} = -\frac{3}{5}\zeta - \left(\frac{3}{5}\right)^3 \mathcal{K}[\zeta^2] + \mathcal{K}_1\left[-\frac{3}{5}\zeta, \left(\frac{3}{5}\right)^3 \mathcal{K}[\zeta^2]\right]\,. \tag{44}$$

Using this iterative procedure we express the kernel $\mathcal{K}$ in Eq. (33) as a function of ζ and we can determine the n-point connected correlation functions for the temperature anisotropies (33) accounting for the information contained in $\mathcal{K}[\zeta]$. In the following we outline the procedure and we give the results, while more details on the computation can be found in the Appendix.

In order to obtain the n-point (connected) correlation functions we will borrow again some techniques of functional-integral analysis from quantum field theory [25, 26, 27] (for different applications of the path-integral approach in cosmology, see Refs. [28, 29, 30, 31, 32, 33]). In particular we need the so called *generating functional of the correlation functions* W, which in our case can be written as

$$Z[J] = \int \mathcal{D}[\zeta]\mathcal{P}[\zeta]e^{i\int d\mathbf{x}J(\mathbf{x})\,(e^{-\zeta/5-\mathcal{K}(\zeta)/5}-1)}\,. \tag{45}$$

Here $J(\mathbf{x})$ is an external source perturbing the underlying statistics, $\mathcal{P}[\zeta]$ is the probability density functional which in our case is a Gaussian one (see Eq. (A.3) of the Appendix), and $\mathcal{D}[\zeta]$ is a suitable measure such that the total probability is normalized to unity, $\int \mathcal{D}[\zeta]\mathcal{P}[\zeta] = 1$. The kernel $\mathcal{K}[\zeta]$ will be given by the iterative procedure. From the generating functional one can obtain the correlation functions by taking the functional derivatives of $Z[J]$ with respect to the source J evaluated at $J = 0$ [26]

$$\begin{aligned}&\langle(e^{-\zeta_1/5-\mathcal{K}(\zeta_1)/5}-1)...(e^{-\zeta_n/5-\mathcal{K}(\zeta_n)/5}-1)\rangle = \\ &i^{-n}\frac{\delta^n Z[J]}{\delta J(\mathbf{x}_1)...\delta J(\mathbf{x}_n)}\bigg|_{(J=0)} \equiv Z^{(n)}(\mathbf{x}_1,...,\mathbf{x}_n)\,,\end{aligned} \tag{46}$$

where $\zeta_i \equiv \zeta(\mathbf{x}_i)$.

By defining the new functional $W[J] = \ln Z[J]$ one can obtain the connected correlation functions by the same operations

$$\begin{aligned}&\langle(e^{-\zeta_1/5-\mathcal{K}(\zeta_1)/5}-1)...(e^{-\zeta_n/5-\mathcal{K}(\zeta_n)/5}-1)\rangle_{\text{conn.}} = \\ &i^{-n}\frac{\delta^n W[J]}{\delta J(\mathbf{x}_1)...\delta J(\mathbf{x}_n)}\bigg|_{(J=0)} \equiv W^{(n)}(\mathbf{x}_1,...,\mathbf{x}_n)\,.\end{aligned} \tag{47}$$

It is clear that the complete knowledge of the statistical properties of the perturbations, *i.e.* the complete knowledge of the correlation functions at all orders, can be achieved if one knows the generating functionals. In turn from the definitions (46) and (47) the correlation functions will appear in a power series expansion of the generating functionals, as

$$\begin{aligned}Z[J] &= 1+\sum_{n=1}^{\infty}\frac{i^n}{n!}\int d\mathbf{x}_1...d\mathbf{x}_n Z^{(n)}(\mathbf{x}_1,...,\mathbf{x}_n)\,J(\mathbf{x}_1)...J(\mathbf{x}_n)\,,\\ W[J] &= \sum_{n=1}^{\infty}\frac{i^n}{n!}\int d\mathbf{x}_1...d\mathbf{x}_n W^{(n)}(\mathbf{x}_1,...,\mathbf{x}_n)\,J(\mathbf{x}_1)...J(\mathbf{x}_n)\,.\end{aligned} \tag{48}$$

To compute the functional derivatives of $W[J]$ one can follow the standard procedure to evaluate the connected correlation functions used in field theory [25]. The computation makes use of a perturbative expansion around some known solution, which corresponds to the connected correlation functions of the free scalar field. In our case, the known solution corresponds to the correlation functions when the kernel $\mathcal{K}$ vanishes, which we have computed previously in Eq. (35). As explained in detail in the Appendix, we perform the expansion around $\mathcal{K} = 0$ using as expansion parameter the *r.m.s* amplitude of the cosmological perturbations themselves, that is $\langle \zeta^2 \rangle^{1/2} \sim 10^{-5}$.

A. The Bispectrum

From the general results presented in the Appendix we provide now the expression for the 3-point connected correlation function. It suffices to expand the kernel $\mathcal{K}$ up to second order. The total kernel can be written as a convolution in configuration space

$$\mathcal{K}(\zeta) = \int d\mathbf{x}_1 d\mathbf{x}_2 K_2(\mathbf{x}-\mathbf{x}_1, \mathbf{x}-\mathbf{x}_2)\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\,, \tag{49}$$

where $K(\mathbf{x}-\mathbf{x}_1, \mathbf{x}-\mathbf{x}_2)$ is the double inverse Fourier transform of the expression

$$\widetilde{K}_2(\mathbf{k}_1, \mathbf{k}_2) = (a_{\rm NL} - 1) + \frac{9}{5}\left[(\mathbf{k}_1 \cdot \mathbf{k}_3)(\mathbf{k}_2 \cdot \mathbf{k}_3)/k^4 - (1/3)(\mathbf{k}_1 \cdot \mathbf{k}_2)/k^2\right]\,, \tag{50}$$

where $k = |\mathbf{k}_3|$ and $\mathbf{k}_3 = -(\mathbf{k}_1 + \mathbf{k}_2)$. In Eq. (49) we have added the constant $a_{\rm NL}$ whose role is to parametrize the primordial non-Gaussianity generated during or after inflation in the various scenarios for the generation of the cosmological perturbations, $\zeta = \zeta_{\rm L} + (a_{\rm NL} - 1) \star \zeta_{\rm L}^2$ (from now on we will remove the subscript "L"). For instance, in single field models of inflation $a_{\rm NL} = 1$ (plus tiny contributions proportional to the deviation from scale invariance), in the curvaton scenarion $a_{\rm NL} = (3/4r) - r/2$, where $r \simeq (\rho_\sigma/\rho)_D$ represents the relative curvaton contribution to the total energy density at curvaton decay [7].

We find

$$\begin{aligned} W^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \;&=\; \langle (e^{-\zeta(\mathbf{x}_1)/5} - 1)(e^{-\zeta(\mathbf{x}_2)/5} - 1)(e^{-\zeta(\mathbf{x}_3)/5} - 1)\rangle_{\rm connected} \\ &-\; 5\sum_p \int d\mathbf{y}_1 d\mathbf{y}_2 K_2(\mathbf{x}_{p_2} - \mathbf{y}_1, \mathbf{x}_{p_2} - \mathbf{y}_2)\Big[\widetilde{w}_2(\mathbf{x}_{p_1}, \mathbf{y}_1) \\ &\times\; \widetilde{w}_2(\mathbf{x}_{p_3}, \mathbf{y}_2) + \frac{1}{2}\widetilde{w}_4(\mathbf{x}_{p_1}, \mathbf{x}_{p_3}, \mathbf{y}_1, \mathbf{y}_2)\Big]\,, \end{aligned} \tag{51}$$

where the sum is over all permutations p_1, p_2, p_3 taking the values $(1, 2, 3)$ and

$$
\begin{aligned}
\widetilde{w}_2(\mathbf{x}_1, \mathbf{x}_2) &\equiv -\frac{1}{5}\langle (e^{-\zeta_1/5} - 1)\zeta_2 \rangle_{\rm conn.} = -\frac{1}{5^2} e^{\langle \zeta^2 \rangle/50} \langle \zeta_1 \zeta_2 \rangle \,, && (52) \\
\widetilde{w}_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &\equiv \frac{1}{25}\langle (e^{-\zeta_1/5} - 1)(e^{-\zeta_2/5} - 1)\zeta_3\zeta_4 \rangle_{\rm conn.} && (53) \\
&= \frac{e^{\langle \zeta^2 \rangle/25}}{3 \times 5^4} \left(e^{\langle \zeta_1 \zeta_2 \rangle/25} - 1 \right) \left(\langle \zeta_1 \zeta_4 \rangle + \langle \zeta_2 \zeta_4 \rangle \right) \left(\langle \zeta_1 \zeta_3 \rangle + \langle \zeta_2 \zeta_3 \rangle \right) + \text{cyclic} \,.
\end{aligned}
$$

Despite the fact that we expanded the kernel up to second-order, the expression (51) becomes exact at any order in perturbation theory in the squezeed limit for which $\mathcal{K}$ tends to zero and for single field models of inflation for which $a_{\rm NL} = 1$. In such a case, the exact three-point correlation function for the temperature anisotropies on large-scales is

$$
\begin{aligned}
W^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) && (54) \\
&+ W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_1) \,,
\end{aligned}
$$

where

$$
\begin{aligned}
W_0^{(2)}(\mathbf{x}_i, \mathbf{x}_j) &\equiv e^{\langle \zeta^2 \rangle/50} \left(e^{\langle \zeta_i \zeta_j \rangle/50} - 1 \right) \equiv \int \frac{d^3k}{(2\pi)^3} \, e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)} P(k) \\
&\simeq \frac{1}{50} \int \frac{d \ln k}{2\pi^2} \, j_0 \left(k \left| \mathbf{x}_i - \mathbf{x}_j \right| \right) \mathcal{P}_\zeta(k) \,, && (55)
\end{aligned}
$$

and $\mathcal{P}_\zeta = A(k_0)^2 (k/k_0)^{n_S - 1}$ is the primordial power spectrum of the comoving curvature perturbation with amplitude A and spectral index n_S. The expression for the exact bispectrum of temperature anisotropies valid at any order in perturbation theory is

$$
\left\langle \frac{\delta_{np} T(\mathbf{k}_1)}{T} \frac{\delta_{np} T(\mathbf{k}_2)}{T} \frac{\delta_{np} T(\mathbf{k}_3)}{T} \right\rangle = (2\pi)^3 \, \delta^{(3)} \left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 \right) \, B \left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \right) \,, \qquad (56)
$$

where

$$
\begin{aligned}
B \left(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \right) &= P(k_1) P(k_2) + P(k_1) P(k_3) + P(k_2) P(k_3) \\
&+ \int \frac{d\mathbf{q}}{(2\pi)^3} \, P \left(\left| \mathbf{q} - \mathbf{k}_1 \right| \right) P \left(\left| \mathbf{q} - \mathbf{k}_2 \right| \right) P \left(\left| \mathbf{q} - \mathbf{k}_3 \right| \right) && (57) \\
&\simeq 2 P(k_1) P(k_2) + \int \frac{d\mathbf{q}}{(2\pi)^3} \, P \left(\left| \mathbf{q} \right| \right) P \left(\left| \mathbf{q} - \mathbf{k}_2 \right| \right) P \left(\left| \mathbf{q} + \mathbf{k}_2 \right| \right) \,, (k_1 \ll k_2, k_3) \,.
\end{aligned}
$$

For generic momenta configurations and for models for which $a_{\rm NL}$ is sizeable, the exponentials present in the first line of Eq. (51) has to be expanded at second-order to consistently match the order of the kernel K

$$e^{-\zeta/5} \simeq 1 - \frac{1}{5}\zeta + \frac{1}{2\times 5^2}\zeta^2 + \mathcal{O}\left(\zeta^3\right) . \tag{58}$$

We immediately recover the expression obtained at second-order in perturbation theory for the non-linearity parameter $f_{\rm NL}$ defined as the coefficient of the gravitational potential Φ expanded at second-order in terms of the linear Gaussian gravitational potential $\Phi_{\rm L} = -\phi_1$, $\Phi = \Phi_{\rm L} + f_{\rm NL} \star (\Phi_{\rm L})^2$, with the convential Sachs-Wolfe effect expressed as $\delta T/T = -(\Phi/3)$ [13]

$$f_{\rm NL} = -\left[\frac{5}{3}(1-a_{\rm NL}) + \frac{1}{6}\right] + \left[3(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_2\cdot\mathbf{k}_3)/k^4 - (\mathbf{k}_1\cdot\mathbf{k}_2)/k^2\right] . \tag{59}$$

It may be worth noticing that the coefficient 1/6 in the last expression is simply the result of the expansion of the exponential (58) expressed in terms of the gravitational potential ϕ_1: $e^{-\zeta/5} = e^{-\Phi_{\rm L}/3} \simeq 1 - \frac{1}{3}\Phi_{\rm L} + \frac{1}{18}\Phi_{\rm L}^2$, which gives the contribution $3\times(1/18) = 1/6$ to $f_{\rm NL}$.

B. The Trispectrum

Let us now follow a similar procedure to obtain the 4-point connected correlation function. In this case the kernel $\mathcal{K}$ appearing in Eq. (33) must be expanded up to third order. In configuration space it can be written as a convolution

$$\begin{aligned}\mathcal{K}(\zeta) \;&=\; \int d\mathbf{x}_1 d\mathbf{x}_2 K_2(\mathbf{x}-\mathbf{x}_1,\mathbf{x}-\mathbf{x}_2)\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2) \\ &+\; \int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 K_3(\mathbf{x}-\mathbf{x}_1,\mathbf{x}-\mathbf{x}_2,\mathbf{x}-\mathbf{x}_3)\zeta(\mathbf{x}_1)\zeta(\mathbf{x}_2)\zeta(\mathbf{x}_3) ,\end{aligned} \tag{60}$$

where K_2 is the kernel defined by Eqs. (49) and Eq. (50), while $K_3(\mathbf{x}-\mathbf{x}_1,\mathbf{x}-\mathbf{x}_2,\mathbf{x}-\mathbf{x}_3)$ is the triple inverse Fourier transform of the expression

$$\widetilde{K_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = (b_{\rm NL}-1) + (a_{\rm NL}-1)\,\mathcal{A}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + \mathcal{C}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) , \tag{61}$$

with

$$\begin{aligned}\mathcal{A}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \;&=\; \frac{6}{5}\left[\frac{\mathbf{k}_1\cdot\mathbf{k}_4(\mathbf{k}_3\cdot\mathbf{k}_4+\mathbf{k}_2\cdot\mathbf{k}_4)+(\mathbf{k}_2\cdot\mathbf{k}_4)(\mathbf{k}_3\cdot\mathbf{k}_4)}{k^4}\right. \\ &-\; \left.\frac{1}{3}\frac{\mathbf{k}_1\cdot(\mathbf{k}_2+\mathbf{k}_3)+\mathbf{k}_2\cdot\mathbf{k}_3}{k^2}\right] ,\end{aligned} \tag{62}$$

$$\begin{aligned}\mathcal{C}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) \;&=\; \frac{54}{25}\frac{(\mathbf{k}_4\cdot\mathbf{k}_3)\left[(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_4\right]}{k^4}\left[\frac{(\mathbf{k}_1\cdot(\mathbf{k}_1+\mathbf{k}_2))\,(\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2))}{|\mathbf{k}_1+\mathbf{k}_2|^4}\right. \\ &-\; \left.\frac{1}{3}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{|\mathbf{k}_1+\mathbf{k}_2|^2}\right] + \text{cycl.} ,\end{aligned} \tag{63}$$

where $k = |\mathbf{k}_4|$ and $\mathbf{k}_4 = -(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$. In Eq. (63) one has to take cyclic terms by an exchange of the wavenumbers $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$.

In order to compute $\widetilde{K_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ we have applied the iterative procedure described in Section IV taking into account also a possible primordial non-Gaussian contribution by expanding the curvature perturbation as

$$\zeta = \zeta_{\rm L} + (a_{\rm NL} - 1) \star \zeta_{\rm L}^2 + (b_{\rm NL} - 1) \star \zeta_{\rm L}^3 \,. \tag{64}$$

The value of $b_{\rm NL}$ will depend on the different scenarios for the generation of the cosmological perturbations. For example, for standard single-field models of inflation $b_{\rm NL} = 1$ (plus tiny contributions proportional to powers of the slow-roll parameters), while for other scenarios it might well be non-negligible. For simplicity from now on we will remove the subscript "L".

Similarly to the bispectrum, also for the 4-point connected correlation function there exists a specific limit for which the kernel $\mathcal{K}$ tends to zero, corresponding to take two wavenumbers much smaller than the other ones, *e.g.* $k_1, k_2 \ll k_3, k_4$. For this limit and in the case of single-field models of inflation ($a_{\rm NL} = 1, b_{\rm NL} = 1$), one can compute an exact expression of the trispectrum by Fourier transforming the 4-point connected correlation function which, in this limit, reads

$$
\begin{aligned}
& W^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \langle (e^{-\zeta(\mathbf{x}_1)/5} - 1)(e^{-\zeta(\mathbf{x}_2)/5} - 1)(e^{-\zeta(\mathbf{x}_3)/5} - 1)(e^{-\zeta(\mathbf{x}_4)/5} - 1) \rangle_{\rm conn.} \\
& = W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \\
& + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \left(W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) \right) \\
& + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \left(W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) + W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) \right) \\
& + W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \left(W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) \right) \\
& + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \left(W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) \right) \\
& + \left(W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) + W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \right) \left(W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) + \right. \\
& \left. + W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) \right) \\
& + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) \left(W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) + W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) + W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \right) \\
& + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \\
& + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_2) W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) \left(W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) + W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) + W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \right) \\
& + W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_4) W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_3) \left(W_0^{(2)}(\mathbf{x}_1, \mathbf{x}_3) + W_0^{(2)}(\mathbf{x}_2, \mathbf{x}_4) + W_0^{(2)}(\mathbf{x}_3, \mathbf{x}_4) \right)
\end{aligned}
$$

$$+W_0^{(2)}(\mathbf{x}_2,\mathbf{x}_4)W_0^{(2)}(\mathbf{x}_3,\mathbf{x}_4)\left(W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_2)+W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_3)+W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_4)\right)$$
$$+\left(W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_2)+W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_3)\right)\left(W_0^{(2)}(\mathbf{x}_2,\mathbf{x}_3)W_0^{(2)}(\mathbf{x}_2,\mathbf{x}_4)+W_0^{(2)}(\mathbf{x}_2,\mathbf{x}_3)W_0^{(2)}(\mathbf{x}_3,\mathbf{x}_4)\right)$$
$$+W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_2)W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_4)W_0^{(2)}(\mathbf{x}_3,\mathbf{x}_4)+W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_3)W_0^{(2)}(\mathbf{x}_1,\mathbf{x}_4)W_0^{(2)}(\mathbf{x}_2,\mathbf{x}_4)\,. \qquad (65)$$

On the other hand, for generic momenta configurations and for models for which $a_{\rm NL}$ and $b_{\rm NL}$ are sizeable we can expand the exponential entering in the expression (33) of the temperature anisotropies up to third order and use the kernel $\mathcal{K}(\zeta)$ in Eq. (60). In this way we are able to determine the non-linearity parameter $g_{\rm NL}$ which enters into the trispectrum of the CMB anisotropies according, for example, to the analysis of Refs. [8, 10]. The parameter $g_{\rm NL}$ is defined through the expansion of the (Bardeen) gravitational potential Φ up to third-order as

$$\Phi=\Phi_L+f_{\rm NL}\star(\Phi_L)^2+g_{\rm NL}\star(\Phi_L)^3\,, \qquad (66)$$

where $\Phi_L=-\phi_1$, is the linear Gaussian part of Φ. We find the following expression

$$\begin{aligned} g_{\rm NL}(\mathbf{k_1},\mathbf{k}_2,\mathbf{k_3}) \;=\; & \frac{25}{9}(b_{\rm NL}-1)+\frac{25}{9}(a_{\rm NL}-1)\mathcal{A}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)+\frac{25}{9}\mathcal{C}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)-\frac{5}{9}(a_{\rm NL}-1)\\ &+\;\frac{1}{54}-\frac{1}{3}\left[\frac{(\mathbf{k}_1\cdot(\mathbf{k}_1+\mathbf{k}_2))\,(\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2))}{|\mathbf{k}_1+\mathbf{k}_2|^4}-\frac{1}{3}\frac{\mathbf{k}_1\cdot\mathbf{k}_2}{|\mathbf{k}_1+\mathbf{k}_2|^2}+\text{cycl.}\right]\,, \end{aligned} \qquad (67)$$

where $\mathcal{A}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ and $\mathcal{C}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ are defined through Eqs. (62) and (63).

V. CONCLUSIONS

In this paper we showed how to calculate exactly the n-point correlation function of CMB anisotropies in the case in which all wavelengths are beyond the horizon at last scattering. In this limit the Sachs-Wolfe effect is predominant and its contribution to higher-order correlation functions yields the most direct signal of non-Gaussianity in the primordial cosmological seeds. Of particular interest are the bispectrum and the trispectrum which can be used to assess the level of non-Gaussianity on cosmological scales. We have calculated the non-perturbative expressions for the bispectrum and the trispecturm as predicted within single-field models of inflation and in the so-called "squeezed" limit in which some of the wavenumbers are much smaller than the others. For other scenarios of generation of the cosmological perturbations, we have provided the non-linearity parameters $f_{\rm NL}$ and $g_{\rm NL}$ entering respectively the theoretical predictions of the bispectrum and trispectrum. Our results for the bispectrum and the trispectrum represent the essential input in order to obtain the predicted angular bispectrum $B_{l_1 l_2 l_3} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle$ and the trispectrum of CMB anisotropies (and higher order correlation functions) according, for example, to the formalism developed in Refs. [34, 35]. In these works it is shown how to compute the angular bispectrum accounting for a non-trivial wavenumber dependence of the non-linearity parameter $f_{\rm NL}(\mathbf{k}_1, \mathbf{k}_2)$. The angular modulation of the quadratic non-linearity predicted by Eq. (59) is currently under investigation [36], adopting the technique of Ref. [34], in order to look for specific signatures of inflationary non-Gaussianity in the CMB. Notice that for a ΛCDM cosmology a late integrated Sachs-Wolfe effect arising from the explicit time dependence of the linear gravitational potential during the late accelerated phase would also give a contribution on large scales. The formalism developed in this paper can be extended to take into account also for this effect. On the other hand on smaller scales there will be other effects contributing to CMB non-Gaussianity such as gravitational lensing, Shapiro time-delays and Rees-Sciama effects produced at the non-linear level. One should be able to compute the angular connected correlation functions induced by these effects by using the techinique developed in Ref. [37] and to distinguish them from the large-scale Sachs-Wolfe effect provided here thanks to their specific angular dependence. Our predictions for the higher-order correlation functions should be compared model by model with the unavoidable contributions from various secondary anisotropies and systematic effects, such as astrophysical foregrounds.

VI. APPENDIX

In this Appendix we will show in detail how to compute the n-point connected correlation functions (53) by making use of the generating functional $w(\mathbf{x_1}, ..., \mathbf{x_n})$ and in particular how to get the result (51).

Let us consider the quantity

$$\delta_T = e^{\varphi + K(\varphi)} - 1\,, \tag{A.1}$$

where φ is a generic Gaussian random field, and $K(\varphi)$ is a generic functional of φ (apart from multiplicative coefficients φ will be identified with the comoving curvature perturbation ζ and K will be given by the kernel $\mathcal{K}$ through the iterative procedure).

The generating functional for the correlated functions of δ_T is given by

$$Z[J] = \int \mathcal{D}[\varphi]\mathcal{P}[\varphi] e^{i\int d\mathbf{x} J(\mathbf{x})(e^{\varphi + K(\varphi)} - 1)}, \tag{A.2}$$

where $J(\mathbf{x})$ is an arbitrary external source. The functional integral is over all the φ configurations weighted by the Gaussian probability density functional

$$\mathcal{P}[\varphi] = \frac{e^{-\frac{1}{2}\int d\mathbf{y} d\mathbf{x} \varphi(\mathbf{y})\mathcal{G}(\mathbf{y},\mathbf{x})\varphi(\mathbf{x})}}{\int \mathcal{D}[\varphi] e^{-\frac{1}{2}\int d\mathbf{y} d\mathbf{x} \varphi(\mathbf{y})\mathcal{G}(\mathbf{y},\mathbf{x})\varphi(\mathbf{x})}}, \tag{A.3}$$

which has been properly normalized in such a way that the total probability equals unity, $\int \mathcal{D}[\varphi]\mathcal{P}[\varphi] = 1$.

To compute the functional derivatives with respect to J we find it convenient to use an additional arbitrary source $\lambda(\mathbf{x})$. We introduce the following generating functional

$$Z[J,\lambda] = \int \mathcal{D}[\varphi]\mathcal{P}[\varphi] e^{i\int d\mathbf{x} J(\mathbf{x})(e^{\varphi + K(\varphi)} - 1)} e^{i\int d\mathbf{x} \lambda(\mathbf{x})\varphi(\mathbf{x})}, \tag{A.4}$$

which reduces to the expression (A.2) when $\lambda = 0$. Functionals of the form in Eq. (A.4) are common in field theory when computing correlation functions of *composite operators* (in which case $J(\mathbf{x})$ represents a "local coupling"; see for example Ref. [26]). The correlation functions generated by $Z[J,\lambda]$ are given by

$$\begin{aligned} &\langle (e^{\varphi(\mathbf{y}_1) + K(\varphi(\mathbf{y}_1))} - 1) ... (e^{\varphi(\mathbf{y}_n) + K(\varphi(\mathbf{y}_n))} - 1)\varphi(\mathbf{x}_1)...\varphi(\mathbf{x}_m)\rangle \\ &= i^{-n+m} \left. \frac{\delta^{n+m} Z[J,\lambda]}{\delta J(\mathbf{y}_1)...\delta J(\mathbf{y}_n)\delta\lambda(\mathbf{x}_1)....\delta\lambda(\mathbf{x}_m)} \right|_{(J,\lambda)=0}. \end{aligned} \tag{A.5}$$

Thus we will take only derivatives with respect to J to get the correlation functions of $e^{\varphi + K(\varphi)}$. If we now write

$$e^{\varphi + K(\varphi)} - 1 = \sum_{n=1}^{\infty} \frac{K^n(\varphi)}{n!} e^{\varphi} + (e^{\varphi} - 1), \tag{A.6}$$

we can rewrite Eq. (A.4) as

$$Z[J,\lambda] = e^{i\int d\mathbf{x} J(\mathbf{x}) \left(\sum_{n=1} \frac{K^n(\frac{1}{i}\frac{\delta}{\delta\lambda})}{n!} e^{\frac{1}{i}\frac{\delta}{\delta\lambda}} \right)} Z_0[J,\lambda], \tag{A.7}$$

where $Z_0[J,\lambda]$ corresponds to the generating functional $Z[J,\lambda]$ when the function $K(\varphi) = 0$

$$\begin{aligned} Z_0[J,\lambda] &= \int \mathcal{D}[\varphi]\mathcal{P}[\varphi] e^{i\int d\mathbf{x} J(\mathbf{x})(e^{\varphi} - 1)} e^{i\int d\mathbf{x} \lambda(\mathbf{x})\varphi(\mathbf{x})} \\ &\equiv e^{W_0[J,\lambda]}. \end{aligned} \tag{A.8}$$

In writing Eq. (A.7) we have made use of the property $i^{-1}\delta e^{i\int d\mathbf{y}\lambda(\mathbf{y})\varphi(\mathbf{x})}/\delta J(\mathbf{y}) = \varphi(\mathbf{x})e^{i\int d\mathbf{y}\lambda(\mathbf{y})\varphi(\mathbf{y})}$, in order to isolate the "interaction term" (see, for example, Ref. [25]).

Now let us write

$$W[J,\lambda] \equiv \ln Z[J,\lambda] = W_0[J,\lambda] + \ln\left[1 + e^{-W_0}\left(e^{i\int d\mathbf{x}J(\mathbf{x})\left(\sum_{n=1}\frac{K^n(\frac{1}{i}\frac{\delta}{\delta\lambda})}{n!}e^{\frac{1}{i}\frac{\delta}{\delta\lambda}}\right)} - 1\right)e^{W_0}\right]. \tag{A.9}$$

The derivatives with respect to J evaluated for $(J,\lambda) = 0$ will give the connected correlation functions we are looking for, accounting also for the kernel K. We are using the standard procedure to evaluate the connected correlation functions, for example, for an interacting scalar field (see e.g. Ref. [25]), except that in our case the "interaction term" is related to the kernel K and we will make a perturbative expansion around $W_0[J,\lambda]$, since the derivatives of $W_0[J,\lambda]$ with respect to J (evaluated for $(J,\lambda) = 0$) give the connected correlated functions for $(e^{\varphi} - 1)$, see Eq. (34).

At this point we have to perform a perturbative expansion. We suppose that the perturbation parameter is the small r.m.s amplitude of the perturbations, $\varphi_{\rm rms} \ll 1$ and use the fact that the function $K(\varphi)$ is obtained through the iterative procedure described in Section III. Let us suppose that

$$K(\varphi) = a * \varphi^2 + b * \varphi^3 + ...\,, \tag{A.10}$$

where the star denotes a convolution operation in configuration space. For the sake of simplicity, in the following we will neglect the convolutions, and treat $a, b, ..$ as constant coefficients. In Eq. (A.9) we will focus on the term $\ln(1+\Delta)$ where the definition of Δ is obtained by comparison with Eq. (A.9). Thus $\ln(1+\Delta) = 1 + \Delta - \Delta^2/2 + \Delta^3/3 + ...$

Now let us consider how to compute, for example, the three-point correlated function given in Eq. (51). At lowest order in our approximation, the only term that in this case is relevant is just Δ and moreover we keep only the first term coming from the expansion of the exponential

$$\Delta \simeq e^{-W_0} i \int d\mathbf{x}J(\mathbf{x})\left(\sum_{n=1}\frac{K^n(\frac{1}{i}\frac{\delta}{\delta\lambda})}{n!}e^{\frac{1}{i}\frac{\delta}{\delta\lambda}}\right)e^{W_0}\,. \tag{A.11}$$

Here we have to expand once more, and looking at Eq. (A.10) we just need for the bispectrum to take $K = a\varphi^2$, and from $(e^{\frac{1}{i}\frac{\delta}{\delta\lambda}} = 1 + \delta/(i\delta\lambda) + ...)$ we just pick up the factor 1..

Thus we are left with

$$\ln(1+\Delta) \;\simeq\; \Delta \simeq iae^{-W_0}\int d\mathbf{x}J(\mathbf{x})\left(\frac{1}{i}\frac{\delta}{\delta\lambda}\right)^2 e^{W_0}$$

$$= i^{-1} a \int d\mathbf{x} J(\mathbf{x}) \left(\frac{\delta W_0}{\delta \lambda(\mathbf{x})} \right)^2 + \frac{\delta^2 W_0}{\delta \lambda^2(\mathbf{x})} \,. \tag{A.12}$$

Therefore the bispectrum is given by

$$\begin{aligned} W^{(3)}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= i^{-3} \frac{\delta^3 W[J, \lambda]}{\delta J(\mathbf{y}_1) \delta J(\mathbf{y}_2) \delta J(\mathbf{y}_3)} \bigg|_{(J,\lambda)=0} \\ &= \langle (e^{\varphi(\mathbf{y}_1)} - 1)(e^{\varphi(\mathbf{y}_2)} - 1)(e^{\varphi(\mathbf{y}_3)} - 1) \rangle_{\text{conn.}} \\ &+ i^{-3} \frac{\delta^3 \Delta[J, \lambda]}{\delta J(\mathbf{y}_1) \delta J(\mathbf{y}_2) \delta J(\mathbf{y}_3)} \bigg|_{(J,\lambda)=0} , \end{aligned} \tag{A.13}$$

where we have to compute the last contribution in Eq. (A.13). We need an expression for $W_0[J, \lambda]$. Since the derivatives of W_0 with respect to J and λ (evaluated in $(J, \lambda) = 0$) give the connected correlation functions for $(e^{\varphi} - 1)$ and φ, we can write

$$W_0[J, \lambda] = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d\mathbf{x}_1 ....d\mathbf{x}_n \widetilde{z}(\mathbf{x}_1 ....\mathbf{x}_n) \widetilde{J}(\mathbf{x}_1) ...\widetilde{J}(\mathbf{x}_n) \,, \tag{A.14}$$

where $\widetilde{J}(\mathbf{x}_i)$ can be either the source $J(\mathbf{x}_i)$ or $\lambda(\mathbf{x}_i)$, and correspondingly $\widetilde{w}_n(\mathbf{x}_1....\mathbf{x}_n)$ are the connected correlation functions. For example $\langle (e^{\varphi(\mathbf{x}_1)} - 1)\varphi(\mathbf{x}_2)\varphi(\mathbf{x}_3) \rangle$ for the choice $J(\mathbf{x}_1)\lambda(\mathbf{x}_2)\lambda(\mathbf{x}_3)$, and one has to consider different combinations.

Using the usual operations for functional derivatives (see, for example, [27]) we find

$$i^{-3} \frac{\delta^3 \Delta[J, \lambda]}{\delta J(\mathbf{y}_1) \delta J(\mathbf{y}_2) \delta J(\mathbf{y}_3)} \bigg|_{(J,\lambda)=0} = a \sum_p \left[\widetilde{w}_2(\mathbf{y}_{p_1}, \mathbf{y}_{p_2} \widetilde{w}_2(\mathbf{y}_{p_3}, \mathbf{y}_{p_2}) + \frac{1}{2} \widetilde{z}_4(\mathbf{y}_{p_1}, \mathbf{y}_{p_2}, \mathbf{y}_{p_3}, \mathbf{y}_{p_3}) \right] \tag{A.15}$$

where the sum is over the permutations p_1, p_2, p_3 of indices $(1, 2, 3)$ and we have used the following notations

$$\begin{aligned} \widetilde{w}_2(\mathbf{x}, \mathbf{y}) &\equiv \langle (e^{\varphi(\mathbf{x})} - 1)\varphi(\mathbf{y}) \rangle_{\text{conn.}} \qquad \text{(A.16)} \\ \widetilde{w}_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}, \mathbf{x}) &\equiv \langle (e^{\varphi(\mathbf{x}_1)} - 1)(e^{\varphi(\mathbf{x}_2)} - 1)\varphi(\mathbf{x})\varphi(\mathbf{x}) \rangle_{\text{conn}} \,. \end{aligned}$$

Notice that it is possible to compute the connected correlation functions appearing in Eq. (A.16) and (A.17) in a similar manner to what one does for e^{φ}.

The result in Eq. (A.15) is a sum of two terms that correspond to the two pieces in Eq. (A.12). One has two evaluate φ at two equal points because we have taken the derivative twice with respect to λ. Finally the product of the two-point connected correlation functions and the presence of the fourth-order connected correlation function are due to the fact that we have the product of two first derivatives $w.r.t\ \lambda$ and a second order derivative $w.r.t\ \lambda$, respectively. Taking then the three derivatives $w.r.t.\ J$ involves $(e^{\varphi}-1)$ in the correlations. Now it is easy to generalize the result (A.15) when a is not a constant coefficient but we have a kernel in configuration space such that

$$K(\varphi)=\int d\bar{\mathbf{x}}_1 d\bar{\mathbf{x}}_2 K(\mathbf{x}-\bar{\mathbf{x}}_{\mathbf{1}},\mathbf{x}-\bar{\mathbf{x}}_{\mathbf{2}})\varphi(\bar{\mathbf{x}}_1)\varphi(\bar{\mathbf{x}}_2)+.... \tag{A.17}$$

As one can guess from Eq. (A.15) one has

$$\begin{aligned}
& i^{-3}\ \frac{\delta^3\Delta[J,\lambda]}{\delta J(\mathbf{y}_1)\delta J(\mathbf{y}_2)\delta J(\mathbf{y}_3)}\bigg|_{(J,\lambda)=0} = \\
& \textstyle\sum_p \displaystyle\int d\bar{\mathbf{x}}_1 d\bar{\mathbf{x}}_2 K(\mathbf{y}_{p_2}-\bar{\mathbf{x}}_{\mathbf{1}},\mathbf{y}_{p_2}-\bar{\mathbf{x}}_{\mathbf{2}})\widetilde{w}_2(\mathbf{y}_{p_1},\bar{\mathbf{x}}_1)\widetilde{w}_2(\mathbf{y}_{p_3},\bar{\mathbf{x}}_2) \\
& +\ \frac{1}{2}\sum_p\int d\bar{\mathbf{x}}_1 d\bar{\mathbf{x}}_2 K(\mathbf{y}_{p_3}-\bar{\mathbf{x}}_{\mathbf{1}},\mathbf{y}_{p_3}-\bar{\mathbf{x}}_{\mathbf{2}})\widetilde{w}_4(\mathbf{y}_{p_1},\mathbf{y}_{p_2},\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2)\,.
\end{aligned} \tag{A.18}$$

In fact we have explicitly verified that this is the correct generalization of Eq. (A.15). Thus from Eq. (A.9) and Eq. (A.18) the connected three-point correlation function for $(e^{\varphi+K(\varphi)}-1)$ reads

$$\begin{aligned}
& W^{(3)}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)=i^{-3}\frac{\delta^3 Z[J,\lambda]}{\delta J(\mathbf{y}_1)\delta J(\mathbf{y}_2)\delta J(\mathbf{y}_3)}\bigg|_{(J,\lambda)=0} = \\
& \langle(e^{\varphi(\mathbf{y}_1)}-1)(e^{\varphi(\mathbf{y}_2)}-1)(e^{\varphi(\mathbf{y}_3)}-1)\rangle_{\rm connected} \\
& +\ \sum_p\int d\bar{\mathbf{x}}_1 d\bar{\mathbf{x}}_2 K(\mathbf{y}_{p_2}-\bar{\mathbf{x}}_{\mathbf{1}},\mathbf{y}_{p_2}-\bar{\mathbf{x}}_{\mathbf{2}})\widetilde{w}_2(\mathbf{y}_{p_1},\bar{\mathbf{x}}_1)\widetilde{w}_2(\mathbf{y}_{p_3},\bar{\mathbf{x}}_2) \\
& +\ \frac{1}{2}\sum_p\int d\bar{\mathbf{x}}_1 d\bar{\mathbf{x}}_2 K(\mathbf{y}_{p_3}-\bar{\mathbf{x}}_{\mathbf{1}},\mathbf{y}_{p_3}-\bar{\mathbf{x}}_{\mathbf{2}})\widetilde{w}_4(\mathbf{y}_{p_1},\mathbf{y}_{p_2},\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2)\,,
\end{aligned} \tag{A.19}$$

where one has to use the definitions in Eqs. (A.16) and (A.17).

Acknowledgments

N.B. would like to thank James Babington for useful discussions on techniques of functional-integral analysis in quantum field theory.

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